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ON A DETERMINATION OF THE COEFFICIENTS OF THE SECOND ORDER HYPERBOLIC EQUATION WITH DISCONTINUOUS SOLUTION

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Abstract. In this paper the problem of finding the coefficients of the second-order hyperbolic equation with discontinuous solutions is studied. The problem under consideration is reduced to the optimal control problem. Existence theorem for an optimal control is proved and necessary optimality condition in the form of a variational inequality is derived.

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1 Introduction

Recently, for solving ill-posed and inverse problems for partial differential equations, the variational or optimization method has been intensively studied and it is widely applied to various problems (Alifanov et al., 1988; Glasko, 1984; Kabanikhin & Iskakov, 2001; Uhlmann & Zhai, 2021). The essence of this method is that the considered ill-posed problem is reduced to the optimal control problem that is studied by methods of optimal control theory (Vasilev, 1981; Krupchyk & Uhlmann, 2020; Tagiev, 2021; Uhlmann & Zhang, 2022; Romanov & Yamamoto, 2019).

It is known that the coefficients of the partial differential equations characterize the properties of the medium under consideration. Therefore, finding the unknown coefficients of the equation is of great importance from both theoretical and practical points of view. In most cases, such problems are ill-posed (Kabanikhin, 2009; Tikhonov & Arsenin, 1974; Valitov & Kozhanov, 2006).

In addition, if the process under consideration is described by the equation with a discontinuous solution, then the study of the boundary value problem and, accordingly, the optimal control problem becomes much more complicated (Ladyzhenskaya, 2013; Serovajsky, 2009). In this paper the problem of finding the coefficients of the second-order hyperbolic equation with discontinuous solutions is studied. The problem is reduced to the optimal control problem (Lions, 1972, 1987; Kabanikhin & Shishlenin, 2012; Yuldashev, 2019). For this problem the existence theorem for the optimal control is proved and a necessary optimality condition in the form of a variational inequality is derived.

2 Problem formulation

Let the controlled process in the domain $Q = \Omega \times (0,\ T)$ be described by the following hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - u^3 + \upsilon_1(x)\frac{\partial u}{\partial x_1} + \upsilon_2(x)\frac{\partial u}{\partial x_2} = f(x,\ t) \tag{1}$$

with initial

$$u(x,\ 0) = u_0(x), \frac{\partial u}{\partial x}(x,\ 0) = u_1(x), x \in \Omega, \tag{2}$$

and boundary conditions

$$u|_{\Sigma} = 0. \tag{3}$$

Here $Q = \Omega \times (0,\ T)$ is a cylinder, $\Omega \subset R^2$ is a bounded domain with smooth boundary Γ, $\Sigma = \Gamma \times (0,\ T)$ is a lateral surface of Q, $f(x,\ t) \in L_2(Q)$, $u_0(x) \in \overset{0}{W}{}_2^1(\Omega)$, $u_1(x) \in L_2(\Omega)$ are given functions, $T > 0$ is a given number.

Let

$$V = \Big\{\upsilon(x) : \upsilon(x) = (\upsilon_1(x), \upsilon_2(x))\,, \upsilon_i(x) \in C^1(\overline{\Omega}), |\upsilon_i(x)| \le \mu_i\ ,$$
$$\left|\frac{\partial \upsilon_i(x)}{\partial x_k}\right| \le \mu_i^k in\Omega, i, k = 1, 2\Big\}, \tag{4}$$

$\mu_i > 0$, $\mu_i^k > 0$ be given numbers.

The pair $\{\upsilon,\ u\}$ is called an admissible pair if $\upsilon \in V$, $u \in L_6(Q)$ and (1)-(3) is satisfied in weak sense, i.e. in the sense of integral identity.

Assume that the set of admissible pairs is not empty i.e.

$$\{\upsilon,\ u\} \ne \emptyset \tag{5}$$

and it needs to minimize the functional

$$J(\upsilon,\ u) = \frac{1}{6}\,\|u - u_d\|^6_{L_6(Q)} + \frac{N}{2}\left(\|\upsilon_1\|^2_{L_2(\Omega)} + \|\upsilon_2\|^2_{L_2(\Omega)}\right) \tag{6}$$

on the set of admissible pairs, where $u_d \in L_6(Q)$ is a given function, $N > 0$ is a given number.

3 Existence of the optimal pair

Theorem 1. *Under the imposed conditions on the problem data, in problem* (1)-(6) *there is an optimal pair* $\{\tilde{\upsilon},\ \tilde{u}\}$, *i.e.*

$$J(\tilde{\upsilon},\ \tilde{u}) = \inf J(\upsilon,\ u), \tag{7}$$

where $\{\upsilon,\ \ u\}$ *is an admissible pair.*

Proof. Let $\left\{\upsilon^k,\ u^k\right\}$ be a minimizing sequence (Lions, 1987) i.e.

$$\lim_{k\to\infty} J(\upsilon^k,\ u^k) = \inf_{\{\upsilon, u\}} J(\upsilon,\ u).$$

As follows from the definition of $J(\upsilon,\ u)$

$$\left\|{\upsilon_1}^k\right\|_{L_2(\Omega)} + \left\|{\upsilon_2}^k\right\|_{L_2(\Omega)} + \left\|u^k\right\|_{L_6(Q)} \le c, \tag{8}$$

here and further on by c we denote various constants that do not depend on the quantities being estimated and on the admissible pair.

Then the sequence $\left\{ {u^k}^3 - {\upsilon_1}^k(x)\frac{\partial u^k}{\partial x_1} - {\upsilon_2}^k(x)\frac{\partial u^k}{\partial x_2} \right\}$ is bounded in the space $L_2(Q)$, therefore from the equation

$$\frac{\partial^2 u^k}{\partial t^2} - \Delta u^k = {u^k}^3 - {\upsilon_1}^k(x)\frac{\partial u^k}{\partial x_1} - {\upsilon_2}^k(x)\frac{\partial u^k}{\partial x_2} + f(x,\ t) \tag{9}$$

follows the estimation

$$\left\| u^k \right\|_{L_\infty(0,\ T;\ \overset{0}{W}{}_2^1(\Omega))} + \left\| \frac{\partial u^k}{\partial t} \right\|_{L_\infty(0,\ T;\ L_2(\Omega))} \leq c. \tag{10}$$

Then one can choose such subsequence from the sequence $\left\{ \upsilon^k,\ u^k \right\}$ that

$$\begin{gathered} {\upsilon_i}^k \to \tilde{\upsilon}_i \ in\ L_2(\Omega)\ weakly,\ \ i = 1,\ 2, \\ u^k \to \tilde{u}\ in\ L_\infty(0,\ T;\ \overset{0}{W}{}_2^1(\Omega)) *\ -weakly, \\ \frac{\partial u^k}{\partial t} \to \frac{\partial \tilde{u}}{\partial t}\ in\ L_\infty(0,\ T;\ L_2(\Omega)) *\ -weakly \end{gathered} \tag{11}$$

hold at $k \to \infty$,

In addition, taking into account the definition of the set V, we can assume that ${\upsilon_i}^k \to \tilde{\upsilon}_i$ in $C(\bar{\Omega})$, $i = 1,\ 2$ at $k \to \infty$.

Then

$$\underline{\lim}_{k\to\infty} J(\upsilon^k,\ u^k) \geq J(\tilde{\upsilon},\ \tilde{u}), \lim_{k\to\infty} J(\upsilon^k,\ u^k) = \inf J(\upsilon,\ u) \Rightarrow$$

$$\inf J(\upsilon,\ u) = \underline{\lim}_{k\to\infty} J(\upsilon^k\ u^k) \geq J(\tilde{\upsilon},\ \tilde{u}).$$

Thus, $\{\tilde{\upsilon},\ \tilde{u}\}$ is an optimal pair. The theorem is proved.

$\square$

4 Convergence of the adapted penalty method

Introduce the functional

$$\begin{aligned} J_\varepsilon^a(\upsilon,\ u) = \tfrac{1}{6}\left\| u - u_d \right\|_{L_6(Q)}^6 + \tfrac{N}{2}\left(\left\| \upsilon_1 \right\|_{L_2(\Omega)}^2 + \left\| \upsilon_2 \right\|_{L_2(\Omega)}^2 \right) + \\ + \tfrac{1}{2\varepsilon}\left\| \tfrac{\partial^2 u}{\partial t^2} - \Delta u - u^3 + \upsilon_1\tfrac{\partial u}{\partial x_1} + \upsilon_2\tfrac{\partial u}{\partial x_2} - f \right\|_{L_2(Q)}^2 + \tfrac{1}{2}\left\| u - \tilde{u} \right\|_{L_2(Q)}^2 + \\ + \tfrac{1}{2}\left\| \upsilon_1 - \tilde{\upsilon}_1 \right\|_{L_2(\Omega)}^2 + \left\| \upsilon_2 - \tilde{\upsilon}_2 \right\|_{L_2(\Omega)}^2 \end{aligned} \tag{12}$$

adopted to the chosen optimal pair $\{\tilde{\upsilon},\ \tilde{u}\}$, where the functions $\upsilon,\ u$ satisfy the conditions

$$\begin{gathered} \upsilon \in V, \\ u \in L_6(Q),\quad u'' - \Delta u \in L_2(Q), \\ u(x,\ 0) = u_0(x), \frac{\partial u}{\partial x}(x,\ 0) = u_1(x), x \in \Omega, \\ u|_\Sigma = 0. \end{gathered} \tag{13}$$

Now consider the problem of minimizing the functional $J_\varepsilon^a(\upsilon,\ u)$ subject to (13). As in Theorem 1, it can be shown that there exists an optimal pair $\{\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon\}$ for this problem.

Theorem 2. *Let* $\{\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon\}$ *be any solution to the problem* $J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) = \inf J_\varepsilon^a(\upsilon,\ u)$. *Then for* $\varepsilon \to 0$

$$\tilde{\upsilon}_{i\varepsilon} \to \tilde{\upsilon}_i \ in\ C(\overline{\Omega}),\ i = 1, 2, \tag{14}$$

$$\tilde{u}_\varepsilon \to \tilde{u} \ in\ L_6(Q), \tag{15}$$

where $\{\tilde{\upsilon},\ \tilde{u}\}$ *is a chosen optimal pair.*

Proof. It is obvious that

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) = \inf J_\varepsilon^a(\upsilon,\ u) \le J_\varepsilon^a(\tilde{\upsilon},\ \tilde{u}) = J(\tilde{\upsilon},\ \tilde{u}). \tag{16}$$

Hence, by the definition of the functional, we obtain

$$\|\tilde{\upsilon}_{1\varepsilon}\|_{L_2(\Omega)} + \|\tilde{\upsilon}_{2\varepsilon}\|_{L_2(\Omega)} + \|\tilde{u}_\varepsilon\|_{L_6(Q)} \le c. \tag{17}$$

where c is a constants not depending on ε. Also it is true

$$\begin{gathered}\frac{\partial^2 \tilde{u}}{\partial t^2} - \Delta\tilde{u}_\varepsilon - \tilde{u}_\varepsilon{}^3 + \tilde{\upsilon}_{1\varepsilon}(x)\frac{\partial \tilde{u}_\varepsilon}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}(x)\frac{\partial \tilde{u}_\varepsilon}{\partial x_2} = \sqrt{\varepsilon} g_\varepsilon + f,\\ \tilde{u}_\varepsilon(x,\ 0) = u_0(x), \frac{\partial \tilde{u}_\varepsilon}{\partial t}(x,\ 0) = u_1(x), \tilde{u}_\varepsilon = 0 \ on\ \Sigma,\end{gathered} \tag{18}$$

where $g_\varepsilon(x,\ t)$ is a function satisfying $\|g_\varepsilon\|_{L_2(Q)} \le c$.

It follows from (17) and (18) that

$$\|\tilde{u}_\varepsilon\|_{L_\infty(0,\ T;\ \overset{0}{W}{}_2^1(\Omega))} \le c, \left\|\frac{\partial \tilde{u}_\varepsilon}{\partial t}\right\|_{L_\infty(0,\ T;\ L_2(\Omega))} \le c. \tag{19}$$

Due to definition of the class V we can assume that $\tilde{\upsilon}_{i\varepsilon} \to \tilde{\upsilon}_i$ strongly in $C(\bar{\Omega})$ $(i = 1,\ 2.)$. Then from $\{\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon\}$ one may choose such subsequence $\left\{\tilde{\tilde{\upsilon}}_\varepsilon,\ \tilde{\tilde{u}}_\varepsilon\right\}$ that

$$\begin{gathered}\tilde{\upsilon}_{i\varepsilon} \to \tilde{\tilde{\upsilon}}_i \ \ strongly\ in\ C(\bar{\Omega}).\\ \tilde{u}_\varepsilon \to \tilde{\tilde{u}} \ \ in\ \ L_\infty(0, T;\ \overset{0}{W}{}_2^1(\Omega)) * -weakly,\\ \tfrac{\partial \tilde{u}_\varepsilon}{\partial t} \to \tfrac{\partial \tilde{\tilde{u}}}{\partial t} \ \ in\ \ L_\infty(0, T;\ L_2(\Omega)) * -weakly.\\ \tilde{u}_\varepsilon \to \tilde{\tilde{u}} \ \ in\ L_6(Q)\ strongly.\end{gathered} \tag{20}$$

Therefore, the following relations hold in the weak sense

$$\frac{\partial^2 \tilde{\tilde{u}}}{\partial t^2} - \Delta\tilde{\tilde{u}} - \tilde{\tilde{u}}^3 + \tilde{\tilde{\upsilon}}_1(x)\frac{\partial \tilde{\tilde{u}}}{\partial x_1} + \tilde{\tilde{\upsilon}}_2(x)\frac{\partial \tilde{\tilde{u}}}{\partial x_2} = f,$$

$$\tilde{\tilde{u}}(x,\ 0) = u_0(x), \frac{\partial \tilde{\tilde{u}}(x,\ 0)}{\partial t} = u_1(x), \tilde{\tilde{u}}\big|_\Sigma = 0.$$

Then the inequality

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) + \frac{1}{2}\left\|\tilde{u} - \tilde{\tilde{u}}\right\|_{L_2(Q)}^2 + \frac{1}{2}\left\|\tilde{\upsilon}_1 - \tilde{\tilde{\upsilon}}_1\right\|_{L_2(\Omega)}^2 + \frac{1}{2}\left\|\tilde{\upsilon}_2 - \tilde{\tilde{\upsilon}}_2\right\|_{L_2(\Omega)}^2$$

leads to the inequality

$$\varliminf_{\varepsilon\to 0} J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \ge J(\tilde{\tilde{\upsilon}},\ \tilde{\tilde{u}}) + \frac{1}{2}\left\|\tilde{u} - \tilde{\tilde{u}}\right\|_{L_2(Q)}^2 + \frac{1}{2}\left\|\tilde{\upsilon}_1 - \tilde{\tilde{\upsilon}}_1\right\|_{L_2(\Omega)}^2 + \frac{1}{2}\left\|\tilde{\upsilon}_2 - \tilde{\tilde{\upsilon}}_2\right\|_{L_2(\Omega)}^2.$$

Since by virtue of (16) $\overline{\lim\limits_{\varepsilon\to 0}} J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \le J(\tilde{\upsilon},\ \tilde{u})$, it follows that $J(\tilde{\tilde{\upsilon}},\ \tilde{\tilde{u}}) \le J(\tilde{\upsilon},\ \tilde{u})$. Therefore $J(\tilde{\tilde{\upsilon}},\ \tilde{\tilde{u}}) = J(\tilde{\upsilon},\ \tilde{u})$. Then

$$\frac{1}{2}\left\|\tilde{u}-\tilde{\tilde{u}}\right\|^2_{L_2(Q)} + \frac{1}{2}\left\|\tilde{\upsilon}_1-\tilde{\tilde{\upsilon}}_1\right\|^2_{L_2(\Omega)} + \frac{1}{2}\left\|\tilde{\upsilon}_2-\tilde{\tilde{\upsilon}}_2\right\|^2_{L_2(\Omega)} = 0.$$

Thus $\tilde{\tilde{\upsilon}}_1 = \tilde{\upsilon}_1$, $\tilde{\tilde{\upsilon}}_2 = \tilde{\upsilon}_2$, $\tilde{\tilde{u}} = \tilde{u}$ and consequent, we obtain convergence without extracting subsequences. The validity of (14) is obtained from this and from (20).

Since

$$J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon)\ in\ \underline{\lim_{\varepsilon\to 0}} J(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \ge J(\tilde{\upsilon},\ \tilde{u}),$$

it is obvious that

$$J(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) \to J(\tilde{\upsilon},\ \tilde{u}),$$

that considering the definition of $J(\upsilon,\ u)$ leads to (15). Theorem is proved.

□

5 Optimality system for the problem with a penalty

Theorem 3. *Let $\{\tilde{\upsilon},\ \tilde{u}\}$ be an optimal pair i.e. be a solution to problem (7). Then these exists a triple $\{\tilde{\upsilon},\ \tilde{u},\ \psi\}$ such that*

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \Delta\tilde{u} - \tilde{u}^3 + \tilde{\upsilon}_1(x)\frac{\partial \tilde{u}}{\partial x_1} + \tilde{\upsilon}_2(x)\frac{\partial \tilde{u}}{\partial x_2} = f(x,\ t) \in Q,$$

$$\tilde{u}(x,\ 0) = u_0(x), \frac{\partial \tilde{u}}{\partial t}(x,\ 0) = u_1(x), \tilde{u}|_\Sigma = 0,$$

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi - 3\tilde{u}^2\psi - \frac{\partial}{\partial x_1}(\tilde{\upsilon}_1\psi) - \frac{\partial}{\partial x_2}(\tilde{\upsilon}_2\psi) = (\tilde{u}-u_d)^5,$$

$$\psi(x,\ T) = \frac{\partial \psi}{\partial t}(x,\ T) = 0\ in\ \Omega,\ \psi|_\Sigma = 0,$$

$$\tilde{u} \in L^\infty(0,\ T;\ W_2^1(\Omega)),\ \frac{\partial \tilde{u}}{\partial t} \in L^\infty(0,\ T;\ L_2(\Omega)),$$

$$\psi \in L^\infty(0,\ T;\ W_2^{\frac{1}{3}}(\Omega)),\ \frac{\partial \psi}{\partial t} \in L^\infty(0,\ T;\ W_2^{-\frac{2}{3}}(\Omega)),$$

and the following vatiational inequality holds

$$\int_Q \left[\left(N\tilde{\upsilon}_1 - \psi\frac{\partial \tilde{u}}{\partial x_1}\right)(\upsilon_1-\tilde{\upsilon}_1) + \left(N\tilde{\upsilon}_2 - \psi\frac{\partial \tilde{u}}{\partial x_2}\right)(\upsilon_2-\tilde{\upsilon}_2)\right] dxdt \ge 0, \forall\upsilon = (\upsilon_1,\ \upsilon_2) \in V.$$

Proof. We now define the necessary conditions for the fact that $\{\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon\}$ is a solution to the problem $J_\varepsilon^a(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon) = \inf J_\varepsilon^a(\upsilon,\ u)$.

As above $\tilde{u}_\varepsilon(x,t)$ wouold be a solution to the following boundary value problem

$$\frac{\partial^2 \tilde{u}_\varepsilon}{\partial t^2} - \Delta\tilde{u}_\varepsilon - \tilde{u}_\varepsilon{}^3 + \tilde{\upsilon}_{1\varepsilon}(x)\frac{\partial \tilde{u}_\varepsilon}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}(x)\frac{\partial \tilde{u}_\varepsilon}{\partial x_2} = \sqrt{\varepsilon}g_\varepsilon + f, \tag{21}$$

$$\tilde{u}_\varepsilon(x,\ 0) = u_0(x), \frac{\partial \tilde{u}_\varepsilon}{\partial t}(x,\ 0) = u_1(x), \tilde{u}_\varepsilon|_\Sigma = 0, \tag{22}$$

where $g_\varepsilon(x,\ t)$ is a such function that $\|g_\varepsilon\|_{L_2(Q)} \leq c$. The necessary conditions for the optimality of the pair $\{\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon\}$ take the form

$$\frac{d}{d\lambda} J^a_\varepsilon\left(\tilde{\upsilon}_\varepsilon,\ \tilde{u}_\varepsilon + \lambda\xi\right)\Big|_{\lambda=0} = 0 \quad \forall\xi \in C^2(\bar{Q}), \tag{23}$$

$\xi(x,\ 0) = 0$, $\frac{\partial\xi(x.0)}{\partial t} = 0$ in Ω, $\xi = 0$ on Σ,

$$\frac{d}{d\lambda} J^a_\varepsilon\left(\tilde{\upsilon}_\varepsilon + \lambda(\upsilon - \tilde{\upsilon}_\varepsilon),\ \tilde{u}_\varepsilon\right)\Big|_{\lambda=0} \geq 0, \forall\upsilon \in V. \tag{24}$$

It follows from (23) that

$$\begin{aligned}
&\frac{1}{\varepsilon}\int\limits_Q \left(\frac{\partial^2\tilde{u}_\varepsilon}{\partial t^2} - \Delta\tilde{u}_\varepsilon - \tilde{u}^3_\varepsilon + \tilde{\upsilon}_{1\varepsilon}\frac{\partial\tilde{u}_\varepsilon}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}\frac{\partial\tilde{u}_\varepsilon}{\partial x_2} - f\right)\times \\
&\times\left(\frac{\partial^2\xi}{\partial t^2} - \Delta\xi - 3\xi\tilde{u}^2_\varepsilon + \tilde{\upsilon}_{1\varepsilon}\frac{\partial\xi}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}\frac{\partial\xi}{\partial x_2}\right)dxdt+ \\
&+\int\limits_Q (\tilde{u}_\varepsilon - u_d)^5\xi dxdt + \int\limits_Q (\tilde{u}_\varepsilon - u)\xi dxdt = 0.
\end{aligned} \tag{25}$$

To write this condition in more simple form we set

$$\psi_\varepsilon = -\frac{1}{\varepsilon}\left(\frac{\partial^2\tilde{u}_\varepsilon}{\partial t^2} - \Delta\tilde{u}_\varepsilon - \tilde{u}_\varepsilon{}^3 + \tilde{\upsilon}_{1\varepsilon}\frac{\partial\tilde{u}_\varepsilon}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}\frac{\partial\tilde{u}_\varepsilon}{\partial x_2} - f\right).$$

Then (25) takes the form

$$\begin{aligned}
&-\int\limits_Q \psi_\varepsilon\left(\frac{\partial^2\xi}{\partial t^2} - \Delta\xi - 3\xi\tilde{u}^2_\varepsilon + \tilde{\upsilon}_{1\varepsilon}\frac{\partial\xi}{\partial x_1} + \tilde{\upsilon}_{2\varepsilon}\frac{\partial\xi}{\partial x_2}\right)dxdt+ \\
&+\int\limits_Q \left(\tilde{u}_\varepsilon - u_d\right)^5\xi dxdt + \int\limits_Q \left(\tilde{u}_\varepsilon - u\right)\xi dxdt = 0
\end{aligned} \tag{26}$$

$\forall\xi$ with $\xi \in C^2(\overline{Q})$, $\quad\xi(x,\ 0) = 0,\quad \frac{\partial\xi(x,\ 0)}{\partial t} = 0$ in Ω, $\xi = 0$ on Σ.
As follows from (24)

$$\begin{aligned}
&N\int\limits_Q [\tilde{\upsilon}_{1\varepsilon}(\upsilon_1 - \tilde{\upsilon}_{1\varepsilon}) + \tilde{\upsilon}_{2\varepsilon}(\upsilon_2 - \tilde{\upsilon}_{2\varepsilon})]dxdt- \\
&-\int\limits_Q \psi_\varepsilon\left[(\upsilon_1 - \tilde{\upsilon}_{1\varepsilon})\frac{\partial u_\varepsilon}{\partial x_1} + (\upsilon_2 - \tilde{\upsilon}_{2\varepsilon})\frac{\partial u_\varepsilon}{\partial x_2}\right]dxdt+ \\
&+\int\limits_Q [(\tilde{\upsilon}_{1\varepsilon} - \tilde{\upsilon}_1)(\upsilon_1 - \tilde{\upsilon}_{1\varepsilon}) + (\tilde{\upsilon}_{2\varepsilon} - \tilde{\upsilon}_2)(\upsilon_2 - \tilde{\upsilon}_{2\varepsilon})]dxdt \geq 0, \forall\upsilon = (\upsilon_1, \upsilon_2) \in V.
\end{aligned} \tag{27}$$

Then (27) implies

$$\int\limits_Q \left[\left(N\tilde{\upsilon}_{1\varepsilon} - \psi_\varepsilon \frac{\partial \tilde{u}_\varepsilon}{\partial x_1}\right)(\upsilon_1 - \tilde{\upsilon}_{1\varepsilon}) + \left(N\tilde{\upsilon}_{2\varepsilon} - \psi_\varepsilon \frac{\partial \tilde{u}_\varepsilon}{\partial x_2}\right)(\upsilon_2 - \tilde{\upsilon}_{2\varepsilon})\right] dxdt+$$
$$+\int\limits_Q [(\tilde{\upsilon}_{1\varepsilon} - \tilde{\upsilon}_1)(\upsilon_1 - \tilde{\upsilon}_{1\varepsilon}) + (\tilde{\upsilon}_{2\varepsilon} - \tilde{\upsilon}_2)(\upsilon_2 - \tilde{\upsilon}_{2\varepsilon})] dxdt \geq 0, \forall \upsilon = (\upsilon_1, \upsilon_2) \in V. \tag{28}$$

Equation (26) means that ψ_ε is a weak solution to the problem

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta\psi_\varepsilon - 3\tilde{u}_\varepsilon{}^2\psi_\varepsilon - \frac{\partial}{\partial x_1}(\tilde{\upsilon}_{1\varepsilon}\psi_\varepsilon) - \frac{\partial}{\partial x_2}(\tilde{\upsilon}_{2\varepsilon}\psi_\varepsilon) = (\tilde{u}_\varepsilon - u_d)^5 + (\tilde{u}_\varepsilon - u)$$
$$\psi_\varepsilon(x,\ T) = \frac{\partial \psi_\varepsilon}{\partial t}(x,\ T) = 0 \ \ in \ \ \Omega, \tag{29}$$
$$\psi_\varepsilon = 0 \ \ on \ \ \Sigma.$$

By virtue of the results of [9], the solution to problem (29) has the estimate

$$\|\psi_\varepsilon\|_{L_\infty(0,\,T;\,W_2^{\frac{1}{3}}(\Omega))} + \left\|\frac{\partial \psi_\varepsilon}{\partial t}\right\|_{L_\infty(0,\,T;W_2^{-\frac{2}{3}}(\Omega))} \leq c.$$

Then passing to limit at $\varepsilon \to 0$ in problem (21), (22), inequality (28) and in problem (29) we obtain the statement of the theorem. The theorem is proved. □

6 Conclusion

In the paper the existence of an optimal pair has been proved for the optimal control problem under consideration, and a necessary optimality condition has been derived in the form of variational inequality

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